

The dependence of the abstract boundary classification on a set of curves II: How the classification changes when the bounded parameter property satisfying set of curves changes

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Abstract

The abstract boundary uses sets of curves with the bounded parameter property (b.p.p.) to classify the elements of the abstract boundary into regular points, singular points, points at infinity and so on. Building on the material of Part one of this two part series, we show how this classification changes when the set of b.p.p. satisfying curves changes.

1 Introduction

A boundary construction in General Relativity is a method to attach ‘ideal’ points to a Lorentzian manifold. The constructions are designed so that the ideal points can be classified into physically motivated classes such as regular points, singular points, points at infinity and so on.

To do this most boundary constructions use, implicitly or explicitly, a set of curves, usually with a particular type of parametrization. For example the g -boundary, [1], relies on incomplete geodesics with affine parameter, the b -boundary, [2], on incomplete curves with generalised affine parameter and the c -boundary, [3], and its modern variants, [4, 5, 6], on endless causal curves.

Papers such as [7, 8] reiterate the point that careful consideration of the set of curves used in a classification of boundary points is needed to get a correct definition of a singularity. Indeed, the issues with giving a consistent physical interpretation, raised by the non-Hausdorff and non- T_1 separation properties of the g -, b - and older c -boundaries, is related to the set of curves used for classification ‘being too big’, e.g. including precompact timelike geodesics. For these boundaries, as the set of curves is also connected to the construction of the boundary points, the inclusion of ‘too many curves’ is part of the root cause of these separation properties, [9, 10]. For example the non-Hausdorff behaviour of the b -boundary is directly related to the existence of inextendible incomplete curves that have more than one limit point, [11, Proposition 8.5.1].

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The abstract boundary, [7], differs from the boundaries mentioned above as its construction does not depend on a set of curves. It does use, however, a set of curves for the physical classification of its elements. This begs the question of what happens to the classification when the set of curves changes. This paper is the second in a series of two papers which answers this question.

A set of curves must satisfy the bounded parameter property (b.p.p.) in order to be used for the classification of abstract boundary points. Unfortunately, a b.p.p. satisfying set may contain curves that do not contribute to the classification of abstract boundary points. As a consequence the standard algebra of sets, \subset, \cup, \cap , does not tell the full story regarding the relationships between b.p.p. satisfying sets, from the point of view of the abstract boundary. See Part I, [12], for details and examples of this. Part I resolved this issue by generalising \subset, \cup, \cap . We use this generalisation to describe the relationships between the b.p.p. satisfying sets that we consider in this paper.

The paper is divided into four sections. This section continues with a brief presentation of the classification of boundary and abstract boundary points. Section 2 presents an alternate definition of the classification of boundary points. This allows us to reduce our analysis from the study of the 15 sets of the classification to the study of 3 sets. This is a substantial simplification. Section 3 shows how the boundary point classification changes when the set of curves changes, while Section 4 shows how the abstract boundary classification changes.

1.1 Preliminary Definitions

We shall only consider manifolds, \mathcal{M} , that are paracompact, Hausdorff, connected, C^∞ -manifolds with a metric, g . Please refer to [12] for the definition of a curve and the bounded parameter property (b.p.p.).

Definition 1 ([7, Definition 9]). *An embedding, $\phi : \mathcal{M} \rightarrow \mathcal{M}_\phi$, of \mathcal{M} is an envelopment if \mathcal{M}_ϕ has the same dimension as \mathcal{M} . Let $\Phi(\mathcal{M})$ be the set of all envelopments of \mathcal{M} .*

Definition 2 ([7, Definition 4] and [12, Definition 7]). *Let $BPP(\mathcal{M})$ be the set of all sets of curves with the b.p.p. That is $BPP(\mathcal{M}) = \{\mathcal{C} : \mathcal{C} \text{ is a set of curves with the b.p.p.}\}$.*

Definition 3 ([7, Definition 14 and 22, Theorem 18]). *Let $B(\mathcal{M})$ be the set of all ordered pairs (ϕ, U) of envelopments ϕ and subsets U of $\partial\phi(\mathcal{M}) = \phi(\mathcal{M}) - \phi(\mathcal{M})$. That is,*

$$B(\mathcal{M}) = \{(\phi, U) : \phi \in \Phi(\mathcal{M}), U \subset \partial\phi(\mathcal{M})\}.$$

An element (ϕ, U) of $B(\mathcal{M})$, or just $U \subset \partial\phi(\mathcal{M})$, is called a boundary set. If $U = \{p\}$ then $(\phi, \{p\})$, or just $p \in \partial\phi(\mathcal{M})$, is called a boundary point.

Define a partial order \triangleright on $B(\mathcal{M})$ by $(\phi, U) \triangleright (\psi, V)$ if and only if for every sequence $\{x_i\}$ in \mathcal{M} , $\{\psi(x_i)\}$ has a limit point in V implies that $\{\phi(x_i)\}$ has a limit point in U . We can construct an equivalence relation \equiv on $B(\mathcal{M})$ by $(\phi, U) \equiv (\psi, V)$ if and only if $(\phi, U) \triangleright (\psi, V)$ and $(\psi, V) \triangleright (\phi, U)$. We denote the equivalence class of (ϕ, U) by $[(\phi, U)]$.

The abstract boundary is the set

$$\mathcal{B}(\mathcal{M}) = \left\{ [(\phi, U)] \in \frac{B(\mathcal{M})}{\equiv} : \exists (\psi, \{p\}) \in [(\phi, U)] \right\}.$$

It is the set of all equivalence classes of $B(\mathcal{M})$ under the equivalence relation \equiv that contain an element $(\psi, \{p\})$ where $p \in \partial\psi(\mathcal{M})$. The elements of the abstract boundary are referred to as abstract boundary points.

The standard algebra of sets, \subset, \cup, \cap , does not respect the b.p.p., [12, Section 2.1]. The first part of this series, [12], addressed this problem by defining a generalisation, $\subset_{\text{b.p.p.}}, \cup_{\text{b.p.p.}}, \cap_{\text{b.p.p.}}$, of the standard algebra of sets over $\text{BPP}(\mathcal{M})$ that also behaves well with respect to the classification. The details of this generalisation necessary for this paper are presented at the beginning of Section 3.

We now present the classification of boundary points. In [7] the definitions below include references to the differentiability of the metrics involved. While the differentiability of regular points, singular points and points at infinity is an important part of the subject we shall not need this here. It is an easy matter to extend the definitions below and the results of the following sections to include references to the differentiability of the boundary points considered.

Definition 4 (Regular Boundary Point, [7, Definition 28]). *A boundary point $p \in \partial\phi(\mathcal{M})$, $\phi \in \Phi(\mathcal{M})$, is said to be regular if there exists $\psi \in \Phi(\mathcal{M})$ such that*

1. $\phi(\mathcal{M}) \cup \{p\} \subset \mathcal{M}_\psi$ and \mathcal{M}_ψ is a regular submanifold of \mathcal{M}_ϕ ,
2. $\psi(x) = \phi(x)$, for all $x \in \mathcal{M}$, and
3. there exists a metric \hat{g} on \mathcal{M}_ψ so that $\hat{g}|_{\psi(\mathcal{M})} = g$.

We make the following definitions;

$$\begin{aligned} \text{Reg}(\phi) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is a regular point}\} \\ \text{Irreg}(\phi) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is not a regular point}\} \\ &= \partial\phi(\mathcal{M}) - \text{Reg}(\phi). \end{aligned}$$

Definition 5 (Approachable and Unapproachable Points, [7, Definition 24]). *Let $\phi \in \Phi(\mathcal{M})$ and \mathcal{C} be a set of curves with the b.p.p. A boundary point $p \in \partial\phi(\mathcal{M})$ is approachable if there exists $\gamma \in \mathcal{C}$ so that p is a limit point of the image of the curve $\phi \circ \gamma$.*

We make the following definitions;

$$\begin{aligned} \text{App}(\phi, \mathcal{C}) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is approachable}\} \\ \text{Nonapp}(\phi, \mathcal{C}) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is unapproachable}\} \\ &= \partial\phi(\mathcal{M}) - \text{App}(\phi, \mathcal{C}). \end{aligned}$$

Definition 6 (Point at Infinity, [7, Definition 31, 34 and 36]). *Let $\phi \in \Phi(\mathcal{M})$ and \mathcal{C} be a set of curves with the b.p.p. A boundary point $p \in \partial\phi(\mathcal{M})$ is said to be a point at infinity if*

1. $p \notin \text{Reg}(\phi)$,
2. $p \in \text{App}(\phi, \mathcal{C})$,
3. For all $\gamma \in \mathcal{C}$, if p is a limit point of $\phi \circ \gamma$ then γ is unbounded.

We make the following definitions;

$$\begin{aligned}
\text{Inf}(\phi, \mathcal{C}) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is a point at infinity}\} \\
\text{RemInf}(\phi, \mathcal{C}) &= \{p \in \text{Inf}(\phi, \mathcal{C}) : \exists(\psi, U) \in \mathcal{B}(M) \text{ with} \\
&\quad U \subset \text{Reg}(\psi) \text{ so that } (\psi, U) \triangleright (\phi, \{p\})\} \\
\text{EssInf}(\phi, \mathcal{C}) &= \text{Inf}(\phi, \mathcal{C}) - \text{RemInf}(\phi, \mathcal{C}) \\
\text{MixInf}(\phi, \mathcal{C}) &= \{p \in \text{EssInf}(\phi, \mathcal{C}) : \exists(\psi, \{q\}) \in \mathcal{B}(M) \text{ with} \\
&\quad q \in \text{Reg}(\psi) \text{ so that } (\phi, \{p\}) \triangleright (\psi, \{q\})\} \\
\text{PureInf}(\phi, \mathcal{C}) &= \text{EssInf}(\phi, \mathcal{C}) - \text{MixInf}(\phi, \mathcal{C})
\end{aligned}$$

Elements of $\text{RemInf}(\phi, \mathcal{C})$ are referred to as removable points at infinity, $\text{EssInf}(\phi, \mathcal{C})$ as essential points at infinity, $\text{MixInf}(\phi, \mathcal{C})$ as mixed points at infinity and $\text{PureInf}(\phi, \mathcal{C})$ as pure points at infinity.

Definition 7 (Singular Point, [7, Definition 37, 40, 41, 42 and 44]). Let $\phi \in \Phi(\mathcal{M})$ and \mathcal{C} be a set of curves with the b.p.p. A boundary point $p \in \partial\phi(\mathcal{M})$ is said to be a singularity if

1. $p \notin \text{Reg}(\phi)$,
2. $p \in \text{App}(\phi, \mathcal{C})$,
3. There exists $\gamma \in \mathcal{C}$ so that p is a limit point of $\phi \circ \gamma$ and γ is bounded.

We can make the following definitions;

$$\begin{aligned}
\text{Sing}(\phi, \mathcal{C}) &= \{p \in \partial\phi(\mathcal{M}) : p \text{ is a singularity}\} \\
\text{NonSing}(\phi, \mathcal{C}) &= \partial\phi(\mathcal{M}) - \text{Sing}(\phi, \mathcal{C}) \\
\text{RemSing}(\phi, \mathcal{C}) &= \{p \in \text{Sing}(\phi, \mathcal{C}) : \exists(\psi, U) \in \mathcal{B}(M) \text{ with} \\
&\quad U \subset \text{NonSing}(\psi, \mathcal{C}) \text{ so that } (\psi, U) \triangleright (\phi, \{p\})\} \\
\text{EssSing}(\phi, \mathcal{C}) &= \text{Sing}(\phi, \mathcal{C}) - \text{RemSing}(\phi, \mathcal{C}) \\
\text{MixSing}(\phi, \mathcal{C}) &= \{p \in \text{EssSing}(\phi, \mathcal{C}) : \exists(\psi, \{q\}) \in \mathcal{B}(M) \text{ with} \\
&\quad q \in \text{Reg}(\psi) \text{ so that } (\phi, \{p\}) \triangleright (\psi, \{q\})\} \\
\text{PureSing}(\phi, \mathcal{C}) &= \text{EssSing}(\phi, \mathcal{C}) - \text{MixSing}(\phi, \mathcal{C})
\end{aligned}$$

Elements of $\text{RemSing}(\phi, \mathcal{C})$ are called removable singularities, $\text{EssSing}(\phi, \mathcal{C})$ are called essential singularities, $\text{MixSing}(\phi, \mathcal{C})$ are called mixed (or directional) singularities, $\text{PureSing}(\phi, \mathcal{C})$ are called pure singularities.

In [7] the properties of the above definitions are explored with respect to the equivalence relation \equiv . Scott and Szekeres show that the following definitions are well defined.

Definition 8 (Approachable and unapproachable abstract boundary points, [7, Section 5]). Let \mathcal{C} be a set of curves with the b.p.p. then we can define

$$\begin{aligned}
\text{APP}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{App}(\phi, \mathcal{C})\} \\
\text{NONAPP}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{Nonapp}(\phi, \mathcal{C})\}
\end{aligned}$$

Elements of $\text{APP}(\mathcal{C})$ are called approachable abstract boundary points and elements of $\text{NONAPP}(\mathcal{C})$ are called unapproachable abstract boundary points.

Definition 9 (Indeterminate abstract boundary points, [7, Section 5]). *Let \mathcal{C} be a set of curves with the b.p.p., then an abstract boundary point $[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M})$ is an indeterminate abstract boundary point if one of the following is true,*

1. $p \in \text{Reg}(\phi)$,
2. $p \in \text{RemInf}(\phi, \mathcal{C})$, or
3. $p \in \text{RemSing}(\phi, \mathcal{C})$.

Let,

$$\text{INDET}(\mathcal{C}) = \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \text{ is an indeterminate abstract boundary point}\}$$

Definition 10 (Abstract boundary points at infinity, [7, Definition 47 and the paragraph after Definition 48]). *Let \mathcal{C} be a set of curves with the b.p.p., then we can make the following definitions*

$$\begin{aligned} \text{INF}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{EssInf}(\phi, \mathcal{C})\} \\ \text{MIXINF}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{MixInf}(\phi, \mathcal{C})\} \\ \text{PUREINF}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{PureInf}(\phi, \mathcal{C})\}. \end{aligned}$$

Elements of $\text{INF}(\mathcal{C})$ are called abstract boundary points at infinity, elements of $\text{MIXINF}(\mathcal{C})$ are called abstract boundary mixed points at infinity and elements of $\text{PUREINF}(\mathcal{C})$ are called abstract boundary pure points at infinity.

Definition 11 (Singular abstract boundary points, [7, Definition 48 and the following paragraph]). *Let \mathcal{C} be a set of curves with the b.p.p., then we can make the following definitions*

$$\begin{aligned} \text{SING}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{EssSing}(\phi, \mathcal{C})\} \\ \text{MIXSING}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{MixSing}(\phi, \mathcal{C})\} \\ \text{PURESING}(\mathcal{C}) &= \{[(\phi, \{p\})] \in \mathcal{B}(\mathcal{M}) : p \in \text{PureSing}(\phi, \mathcal{C})\}. \end{aligned}$$

Elements of $\text{SING}(\mathcal{C})$ are called abstract boundary singular points, elements of $\text{MIXSING}(\mathcal{C})$ are called abstract boundary mixed (or directional) singularities and elements of $\text{PURESING}(\mathcal{C})$ are called abstract boundary pure singularities.

2 Alternate definitions of the classes of boundary points

Before we study how the classification given above changes with respect to changes in the b.p.p. satisfying set of curves, we revisit the definitions of the classes. We generalise a few of the concepts behind the classification presented in Section 1.1 and express each of the classes of the classification as a union / intersection of more ‘primitive’ sets. This will allow us to reduce the study of the 15 sets of the boundary point classification to the study of 3 sets.

In [12, Definition 6] the following subdivision of $\text{App}(\phi, \mathcal{C})$, $\phi \in \Phi(\mathcal{M})$, $\mathcal{C} \in \text{BPP}(\mathcal{M})$ was introduced.

Definition 12 ([12, Definition 6]). Let $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and $\text{App}_{\text{Inf}}(\phi, \mathcal{C})$ be defined by,

$$\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \{p \in \text{App}(\phi, \mathcal{C}) : \text{there exists } \gamma \in \mathcal{C}, \text{ bounded,} \\ \text{so that } p \text{ is a limit point of } \phi \circ \gamma\}$$

$$\text{App}_{\text{Inf}}(\phi, \mathcal{C}) = \{p \in \text{App}(\phi, \mathcal{C}) : \text{for all } \gamma \in \mathcal{C}, \text{ if } p \text{ is a} \\ \text{limit point of } \phi \circ \gamma \text{ then } \gamma \text{ has unbounded parameter}\}.$$

This generalises the idea of ‘singular point’ and ‘point at infinity’. We can generalise the concepts of ‘mixed’ and ‘pure’ points.

Definition 13. Let $\phi \in \Phi(\mathcal{M})$. A boundary point $p \in \partial\phi(\mathcal{M})$ is mixed if there exists $(\psi, \{q\}) \in \text{B}(\mathcal{M})$ such that

1. $(\phi, \{p\}) \triangleright (\psi, \{q\})$,
2. $q \in \text{Reg}(\psi)$.

If p is not mixed then we shall say that it is a pure boundary point.

We can make the following definitions;

$$\text{Mix}(\phi) = \{p \in \partial\phi(\mathcal{M}) : p \text{ is a mixed boundary point}\} \\ \text{Pure}(\phi) = \partial\phi(\mathcal{M}) - \text{Mix}(\phi).$$

As will become clear, the analysis of how the classification changes would be much easier if we could also define a ‘removable’ point independently of points at infinity and singular points. Unfortunately the small difference in the definition of $\text{RemInf}(\phi, \mathcal{C})$ and $\text{RemSing}(\phi, \mathcal{C})$ is a serious (and probably fatal) impediment to this. If $p \in \text{RemInf}(\phi, \mathcal{C})$ then there must exist $(\psi, U) \in \text{B}(\mathcal{M})$ so that $U \subset \text{Reg}(\psi)$ and $(\psi, U) \triangleright (\phi, \{p\})$. If $p \in \text{RemSing}(\phi, \mathcal{C})$ then there must exist $(\psi, U) \in \text{B}(\mathcal{M})$ so that $U \subset \text{NonSing}(\phi, \mathcal{C})$ and $(\psi, U) \triangleright (\phi, \{p\})$. Thus the definition of a removable point at infinity depends on $\text{Reg}(\phi)$, while the definition of a removable singularity depends on $\text{NonSing}(\phi, \mathcal{C})$. Hence, while the two definitions are similar, to provide a single definition of a removable point we would need to show something like $p \in \text{RemSing}(\phi, \mathcal{C})$ if and only if there exists $(\psi, U) \in \text{B}(\mathcal{M})$ so that $U \subset \text{Reg}(\phi)$ and $(\psi, U) \triangleright (\phi, \{p\})$. Such a result is, almost certainly, false due to the existence of non-approachable boundary points, see the proof of [7, Theorem 43].

We avoid this issue by differentiating between the two ‘types’ of removable point.

Definition 14. Let $\text{Rem}_{\text{Inf}}(\phi)$ be defined as

$$\text{Rem}_{\text{Inf}}(\phi) = \{p \in \partial\phi(\mathcal{M}) : \exists(\psi, U) \in \text{B}(\mathcal{M}) \text{ so that} \\ U \subset \text{Reg}(\phi) \text{ and } (\psi, U) \triangleright (\phi, \{p\})\}.$$

This is the set of all points that are removable in the sense of the definition of a removable point at infinity. Let $\text{Ess}_{\text{Inf}}(\phi) = \partial\phi(\mathcal{M}) - \text{Rem}_{\text{Inf}}(\phi)$.

Definition 15. Let $\text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$ be defined as,

$$\text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) = \{p \in \partial\phi(\mathcal{M}) : \exists(\psi, U) \in \text{B}(\mathcal{M}) \\ \text{so that } U \subset \text{NonSing}(\phi, \mathcal{C}) \text{ and } (\psi, U) \triangleright (\phi, \{p\})\}$$

This is the set of all points that are removable in the sense of the definition of a removable singularity. Let $\text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) = \partial\phi(\mathcal{M}) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$.

Since $\text{Reg}(\phi) \subset \text{NonSing}(\phi, \mathcal{C})$ we see that $\text{Rem}_{\text{Inf}}(\phi) \subset \text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$, for all $\phi \in \Phi(\mathcal{M})$ and $\mathcal{C} \in \text{BPP}(\mathcal{M})$.

These definitions will eventually lead to some interesting results. In particular, when expanding a b.p.p. satisfying set to include more curves it is possible for a pure point at infinity to become a removable singularity.

The classes defined in Section 1.1 can be expressed in terms of the sets we have just given.

Proposition 16. *Let $\phi \in \Phi(\mathcal{M})$ and \mathcal{C} be a set of curves with the b.p.p. Then we have that,*

$$\begin{aligned} \text{Inf}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \\ \text{RemInf}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Inf}}(\phi) \\ \text{EssInf}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Inf}}(\phi) \\ \text{MixInf}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Inf}}(\phi) \cap \text{Mix}(\phi) \\ \text{PureInf}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Inf}}(\phi) \cap \text{Pure}(\phi) \end{aligned}$$

$$\begin{aligned} \text{Sing}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \\ \text{NonSing}(\phi, \mathcal{C}) &= \text{Reg}(\phi) \cup \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cup \text{Nonapp}(\phi, \mathcal{C}) \\ \text{RemSing}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) \\ \text{EssSing}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \\ \text{MixSing}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Mix}(\phi) \\ \text{PureSing}(\phi, \mathcal{C}) &= \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Pure}(\phi). \end{aligned}$$

Proof. The proofs of these statements follow immediately from the definitions, except for the equation $\text{NonSing}(\phi, \mathcal{C}) = \text{Reg}(\phi) \cup \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cup \text{Nonapp}(\phi, \mathcal{C})$ which requires use of the equation $A - (B \cap C) = (A - B) \cup (A - C)$. \square

As $\text{App}_{\text{Inf}}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{C}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and $\text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) = \partial\phi(\mathcal{M}) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$, Proposition 16 implies that we need only study how the three sets $\text{App}(\phi, \mathcal{C})$, $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and $\text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$ change under changes of \mathcal{C} in order to work out how each of the 15 sets of the boundary point classification change.

3 How the boundary classification changes as we change \mathcal{C}

It was claimed in the introduction, Section 1.1, that the algebra of sets $\subset_{\text{b.p.p.}}$, $\cup_{\text{b.p.p.}}$, $\cap_{\text{b.p.p.}}$, defined on $\text{BPP}(\mathcal{M})$, behaves nicely with respect to the classification. By this we

mean the following [12, Section 4], if $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ and $\phi \in \Phi(\mathcal{M})$ then

$$\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D} \Rightarrow \begin{cases} \text{App}(\phi, \mathcal{C}) \subset \text{App}(\phi, \mathcal{D}) \\ \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \subset \text{App}_{\text{Sing}}(\phi, \mathcal{D}) \end{cases}$$

$$\begin{aligned} \text{App}(\phi, \mathcal{C}), \text{App}(\phi, \mathcal{D}) &\subset \text{App}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}) \\ \text{App}_{\text{Sing}}(\phi, \mathcal{C}), \text{App}_{\text{Sing}}(\phi, \mathcal{D}) &\subset \text{App}_{\text{Sing}}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}) \end{aligned}$$

$$\begin{aligned} \text{App}(\phi, \mathcal{C} \cap_{\text{b.p.p.}} \mathcal{D}) &\subset \text{App}(\phi, \mathcal{C}), \text{App}(\phi, \mathcal{D}) \\ \text{App}_{\text{Sing}}(\phi, \mathcal{C} \cap_{\text{b.p.p.}} \mathcal{D}) &\subset \text{App}_{\text{Sing}}(\phi, \mathcal{C}), \text{App}_{\text{Sing}}(\phi, \mathcal{D}). \end{aligned}$$

Note that $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ does not always exist due to the possible existence of curves in \mathcal{C} and \mathcal{D} that have the same image where one is bounded and the other unbounded. See [12, Section 2.1] for details of this. The equations, given above, involving $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ are only valid when $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ exists.

Below it is shown how the boundary point classification differs between;

- \mathcal{C} and \mathcal{D} when $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$
- \mathcal{C}, \mathcal{D} and $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$, when $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ can be defined.

We do not show how the classification differs between \mathcal{C}, \mathcal{D} and $\mathcal{C} \cap_{\text{b.p.p.}} \mathcal{D}$ as it is very unlikely that one would want to restrict the set of curves used for analysis of the boundary. Moreover the details for this case can be determined by following the pattern of results established by the cases $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ and $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$.

3.1 Subset

Given $\mathcal{C} \in \text{BPP}(\mathcal{M})$ we investigate how the boundary classification induced by \mathcal{C} relates to the boundary classification induced by $\mathcal{D} \in \text{BPP}(\mathcal{M})$ when $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. Because of the results of Section 2, we first focus on the sets $\text{App}(\phi, \mathcal{C})$, $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and $\text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$.

Proposition 17. *Let $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ so that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ then for all $\phi \in \Phi(\mathcal{M})$,*

$$\begin{aligned} \text{App}(\phi, \mathcal{C}) &\subset \text{App}(\phi, \mathcal{D}), \\ \text{App}_{\text{Sing}}(\phi, \mathcal{C}) &\subset \text{App}_{\text{Sing}}(\phi, \mathcal{D}), \\ \text{Rem}_{\text{Sing}}(\phi, \mathcal{D}) &\subset \text{Rem}_{\text{Sing}}(\phi, \mathcal{C}). \end{aligned}$$

Proof. The first two statements follow from Proposition 30 of [12].

From the second statement we know that $\partial\phi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, \mathcal{D}) \subset \partial\phi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C})$. Thus, from Definition 7 and as $A - (B \cap C) = (A - B) \cup (A - C)$, we have that, for all $\phi \in \Phi(\mathcal{M})$,

$$\begin{aligned} \text{NonSing}(\phi, \mathcal{D}) &= (\partial\phi(\mathcal{M}) - \text{Irreg}(\phi)) \cup (\partial\phi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, \mathcal{D})) \\ &\subset (\partial\phi(\mathcal{M}) - \text{Irreg}(\phi)) \cup (\partial\phi(\mathcal{M}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C})) \\ &= \text{NonSing}(\phi, \mathcal{C}). \end{aligned}$$

Let $p \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then there exists $(\psi, U) \in \text{B}(\mathcal{M})$ so that $U \subset \text{NonSing}(\psi, \mathcal{D})$ and $(\psi, U) \triangleright (\phi, \{p\})$. From above we know that $U \subset \text{NonSing}(\psi, \mathcal{C})$ and therefore $p \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$, as required. \square

Using Proposition 17 we can determine how the other sets with a dependence on \mathcal{C} appearing in the left hand side of the equations of Proposition 16 are affected.

Corollary 18. *Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ so that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ then for all $\phi \in \Phi(\mathcal{M})$,*

1. $\text{App}(\phi, \mathcal{D}) - \text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D}) \cap \text{Nonapp}(\phi, \mathcal{C}),$
2. $\text{App}_{\text{Sing}}(\phi, \mathcal{D}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D}) \cap (\text{Nonapp}(\phi, \mathcal{C}) \cup \text{App}_{\text{Inf}}(\phi, \mathcal{C})),$
3. $\text{App}_{\text{Inf}}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{App}_{\text{Inf}}(\phi, \mathcal{C}),$
4. $\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - (\text{App}_{\text{Inf}}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{D})$
 $= \text{App}_{\text{Inf}}(\phi, \mathcal{C}) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{D}),$
5. *If $x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then for all $(\psi, U) \in \mathcal{B}(\mathcal{M})$ so that $(\psi, U) \triangleright (\phi, \{x\})$ we have that $U \cap \text{Irreg}(\phi) \cap (\text{App}_{\text{Sing}}(\psi, \mathcal{D}) - \text{App}_{\text{Sing}}(\psi, \mathcal{C})) \neq \emptyset,$*
6. $\text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \subset \text{Ess}_{\text{Sing}}(\phi, \mathcal{D}),$
7. $\text{Ess}_{\text{Sing}}(\phi, \mathcal{D}) - \text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) = \text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{D}).$

Proof. The proofs of (1), (2), (3), (4) and (6) follow directly from Proposition 17 and the definitions. We include them here for completeness. The proof of item (5) follows from Proposition 16 and the definition of $\text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$. The proof of item (7) follows from the definition of $\text{Ess}_{\text{Sing}}(\phi, \mathcal{C})$ and the standard set relations $A - (B - C) = (A \cap C) \cup (A - B)$ and $(A - B) \cap C = A \cap (C - B)$. \square

Figures 1 and 2 give a graphic representation of Proposition 17 and Corollary 18. Proposition 17 and Corollary 18 now let us determine how the boundary point classification itself changes.

Corollary 19. *Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ so that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ then for all $\phi \in \Phi(\mathcal{M})$,*

1. $\text{Inf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{Inf}(\phi, \mathcal{C}).$
 $\text{Inf}(\phi, \mathcal{C}) - (\text{Inf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{Inf}(\phi, \mathcal{C}) - \text{Inf}(\phi, \mathcal{D})$
 $= \text{Irreg}(\phi) \cap (\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{D})).$
2. $\text{RemInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{RemInf}(\phi, \mathcal{C}).$
 $\text{RemInf}(\phi, \mathcal{C}) - (\text{RemInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{RemInf}(\phi, \mathcal{C}) - \text{RemInf}(\phi, \mathcal{D})$
 $= \text{Irreg}(\phi) \cap \text{Rem}_{\text{Inf}}(\phi) \cap (\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{D})).$
3. $\text{EssInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{EssInf}(\phi, \mathcal{C}).$
 $\text{EssInf}(\phi, \mathcal{C}) - (\text{EssInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{EssInf}(\phi, \mathcal{C}) - \text{EssInf}(\phi, \mathcal{D})$
 $= \text{Irreg}(\phi) \cap \text{Ess}_{\text{Inf}}(\phi) \cap (\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{D})).$
4. $\text{MixInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{MixInf}(\phi, \mathcal{C}).$
 $\text{MixInf}(\phi, \mathcal{C}) - (\text{MixInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{MixInf}(\phi, \mathcal{C}) - \text{MixInf}(\phi, \mathcal{D})$
 $= \text{Irreg}(\phi) \cap \text{Ess}_{\text{Inf}}(\phi) \cap \text{Mix}(\phi) \cap (\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Inf}}(\phi, \mathcal{D})).$

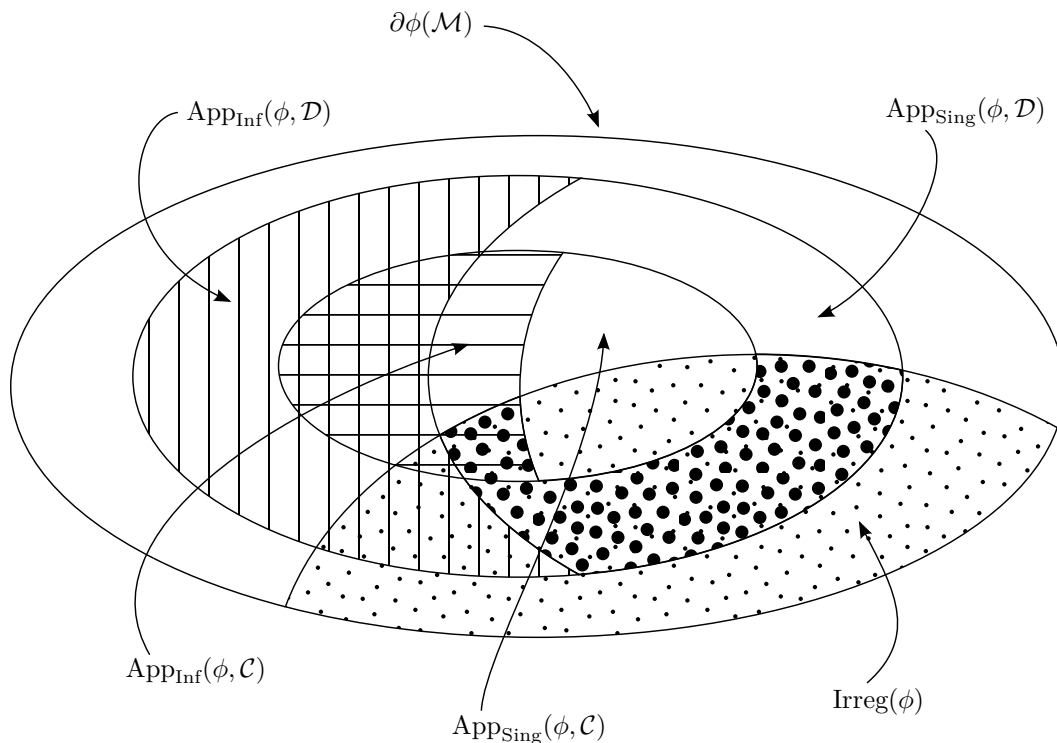


Figure 1: A venn diagram of $\partial\phi(\mathcal{M})$, $\text{Reg}(\phi)$, $\text{Irreg}(\phi)$, $\text{App}(\phi, \mathcal{C})$, $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$, $\text{App}_{\text{Inf}}(\phi, \mathcal{C})$, $\text{App}(\phi, \mathcal{D})$, $\text{App}_{\text{Sing}}(\phi, \mathcal{D})$ and $\text{App}_{\text{Inf}}(\phi, \mathcal{D})$. The outer oval is $\partial\phi(\mathcal{M})$. The dotted region, both small and large dots, is $\text{Irreg}(\phi)$. The middle oval is $\text{App}(\phi, \mathcal{D})$, the inner oval is $\text{App}(\phi, \mathcal{C})$. The region ruled by vertical lines is $\text{App}_{\text{Inf}}(\phi, \mathcal{D})$, the region of the middle oval not ruled by vertical lines is $\text{App}_{\text{Sing}}(\phi, \mathcal{D})$. The region ruled by horizontal lines is $\text{App}_{\text{Inf}}(\phi, \mathcal{C})$, the region of the inner oval not ruled by horizontal lines is $\text{App}_{\text{Sing}}(\phi, \mathcal{C})$. The region ruled by horizontal lines and not ruled by vertical lines is described by item (4) of Corollary 18. The region covered by big dots is $\text{Irreg}(\phi) \cap (\text{App}_{\text{Sing}}(\phi, \mathcal{D}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C}))$, see item (5) of Corollary 18.

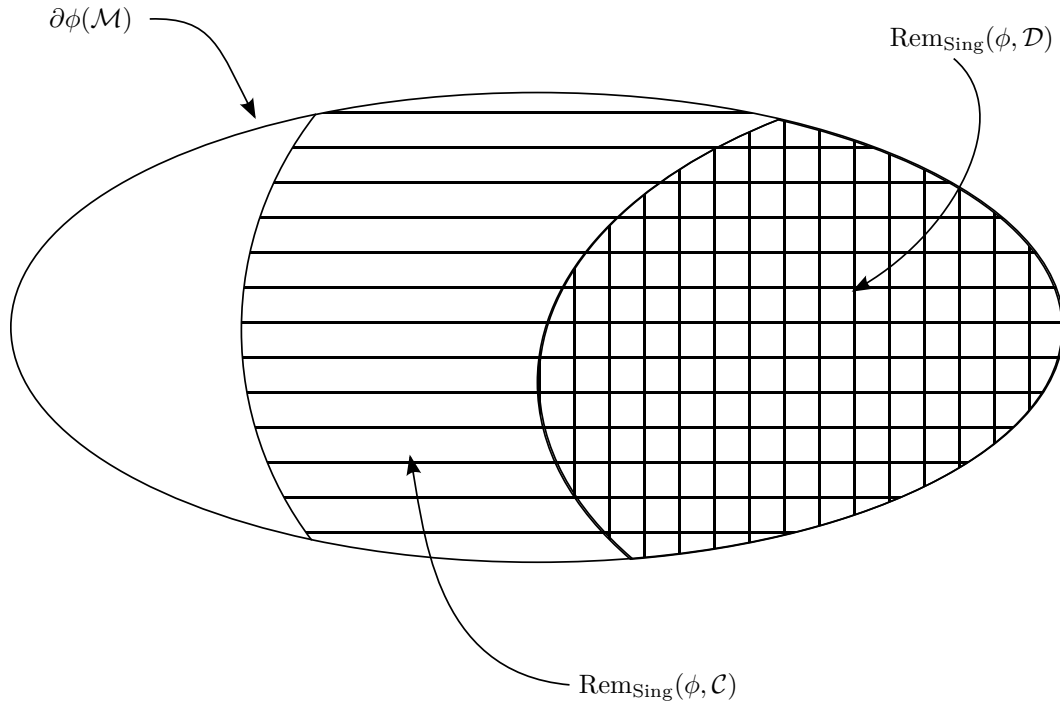


Figure 2: A venn diagram of $\partial\phi(\mathcal{M})$, $\text{RemSing}(\phi, \mathcal{C})$, $\text{EssSing}(\phi, \mathcal{C})$, $\text{RemSing}(\phi, \mathcal{D})$ and $\text{EssSing}(\phi, \mathcal{D})$. The oval is $\partial\phi(\mathcal{M})$. The region ruled by horizontal lines is $\text{RemSing}(\phi, \mathcal{C})$, the region not ruled by horizontal lines is $\text{EssSing}(\phi, \mathcal{C})$. The region ruled by vertical lines is $\text{RemSing}(\phi, \mathcal{D})$, the region not ruled by vertical lines is $\text{EssSing}(\phi, \mathcal{D})$. The region of the oval that is ruled by horizontal lines and not ruled by vertical lines is described by item (5) of Corollary 18, see Figure 1.

5. $\text{PureInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C}) \subset \text{PureInf}(\phi, \mathcal{C})$.

$$\text{PureInf}(\phi, \mathcal{C}) - (\text{PureInf}(\phi, \mathcal{D}) \cap \text{App}(\phi, \mathcal{C})) = \text{PureInf}(\phi, \mathcal{C}) - \text{PureInf}(\phi, \mathcal{D})$$

$$= \text{Irreg}(\phi) \cap \text{EssInf}(\phi) \cap \text{Pure}(\phi) \cap (\text{AppInf}(\phi, \mathcal{C}) - \text{AppInf}(\phi, \mathcal{D})).$$
6. $\text{Sing}(\phi, \mathcal{C}) \subset \text{Sing}(\phi, \mathcal{D})$

$$\text{Sing}(\phi, \mathcal{D}) - \text{Sing}(\phi, \mathcal{C}) = \text{Irreg}(\phi) \cap (\text{AppSing}(\phi, \mathcal{D}) - \text{AppSing}(\phi, \mathcal{C})).$$
7. $\text{NonSing}(\phi, \mathcal{D}) \subset \text{NonSing}(\phi, \mathcal{C})$.

$$\text{NonSing}(\phi, \mathcal{C}) - \text{NonSing}(\phi, \mathcal{D}) = \text{Sing}(\phi, \mathcal{D}) - \text{Sing}(\phi, \mathcal{C}).$$
8. $\text{EssSing}(\phi, \mathcal{C}) \subset \text{EssSing}(\phi, \mathcal{D})$.

$$\text{EssSing}(\phi, \mathcal{D}) - \text{EssSing}(\phi, \mathcal{C}) = \text{EssSing}(\phi, \mathcal{D}) \cap$$

$$\left((\text{AppSing}(\phi, \mathcal{D}) - \text{AppSing}(\phi, \mathcal{C})) \cup (\text{EssSing}(\phi, \mathcal{D}) - \text{EssSing}(\phi, \mathcal{C})) \right).$$
9. $\text{MixSing}(\phi, \mathcal{C}) \subset \text{MixSing}(\phi, \mathcal{D})$

$$\text{MixSing}(\phi, \mathcal{D}) - \text{MixSing}(\phi, \mathcal{C})$$

$$= \text{Mix}(\phi) \cap (\text{EssSing}(\phi, \mathcal{D}) - \text{EssSing}(\phi, \mathcal{C})).$$
10. $\text{PureSing}(\phi, \mathcal{C}) \subset \text{PureSing}(\phi, \mathcal{D})$

$$\text{PureSing}(\phi, \mathcal{D}) - \text{PureSing}(\phi, \mathcal{C})$$

$$= \text{Pure}(\phi) \cap (\text{EssSing}(\phi, \mathcal{D}) - \text{EssSing}(\phi, \mathcal{C})).$$
11. $\text{RemSing}(\phi, \mathcal{D}) \cap \text{Sing}(\phi, \mathcal{C}) \subset \text{RemSing}(\phi, \mathcal{C})$

$$\text{RemSing}(\phi, \mathcal{D}) \cap \text{RemSing}(\phi, \mathcal{C})$$

$$= \text{Irreg}(\phi) \cap \text{AppSing}(\phi, \mathcal{C}) \cap \text{RemSing}(\phi, \mathcal{D}).$$

$$\text{RemSing}(\phi, \mathcal{C}) - (\text{RemSing}(\phi, \mathcal{D}) \cap \text{Sing}(\phi, \mathcal{C})) = \text{RemSing}(\phi, \mathcal{C}) - \text{RemSing}(\phi, \mathcal{D})$$

$$= \text{Irreg}(\phi) \cap \text{AppSing}(\phi, \mathcal{C}) \cap (\text{RemSing}(\phi, \mathcal{C}) - \text{RemSing}(\phi, \mathcal{D})).$$

$$\text{RemSing}(\phi, \mathcal{D}) - \text{RemSing}(\phi, \mathcal{C})$$

$$= \text{Irreg}(\phi) \cap \text{RemSing}(\phi, \mathcal{D}) \cap (\text{AppSing}(\phi, \mathcal{D}) - \text{AppSing}(\phi, \mathcal{C})).$$

Membership of $\text{AppInf}(\phi, \mathcal{C}) - \text{AppInf}(\phi, \mathcal{D})$, $\text{AppSing}(\phi, \mathcal{D}) - \text{AppSing}(\phi, \mathcal{C})$, $\text{EssSing}(\phi, \mathcal{D}) - \text{EssSing}(\phi, \mathcal{C})$ and $\text{RemSing}(\phi, \mathcal{C}) - \text{RemSing}(\phi, \mathcal{D})$ can be checked using Corollary 18.

Proof. Items (1), (2), (3), (4) and (5) follow directly from Proposition 16 and Corollary 18. Item (6) follows from Propositions 16 and 17. The first half of item (7) was proven during the proof of Proposition 16. We include it here for completeness. The second half follows from Definition 7 and as $A - (B - C) = (A \cap C) \cup (A - B)$ and $(A - B) \cap C = A \cap (C - B)$. Item (8) follows from Proposition 16, as $(A \cap B) - (A \cap C) = A \cap (B - C)$, $A - (B \cap C) = (A - B) \cup (A - C)$, $(B \cap A) - C = B \cap (A - C)$ and the distribution law for \cap and \cup . Items (9) and (10) follow from Proposition 16 and the proof of item (8). Item (11) follows from Propositions 16 and 17 and the set relations given above. \square

Corollary 19 gives the relationships between the sets making up the boundary classifications with respect to \mathcal{C} and \mathcal{D} . We can rewrite the results above to emphasise the behaviour of individual boundary points. Since this is the form of the results that is likely to be the most useful, we denote it as a theorem.

Theorem 20. *Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ so that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ then for all $\phi \in \Phi(\mathcal{M})$ we have the following results.*

1. *If $x \in \text{App}(\phi, \mathcal{C})$ then $x \in \text{App}(\phi, \mathcal{D})$.*
2. *If $x \in \text{Nonapp}(\phi, \mathcal{C})$ then either $x \in \text{Nonapp}(\phi, \mathcal{D})$ or $x \in \text{App}(\phi, \mathcal{D})$.*
3. *If $x \in \text{Inf}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Inf}(\phi, \mathcal{D})$*
 - (b) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{D})$*
4. *If $x \in \text{RemInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{RemInf}(\phi, \mathcal{D})$*
 - (b) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{D})$*
5. *If $x \in \text{EssInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{EssInf}(\phi, \mathcal{D})$*
 - (b) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{C})$ and*
 - i. *$x \in \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{D})$.*
 - ii. *$x \notin \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{EssSing}(\phi, \mathcal{D})$.*
6. *If $x \in \text{MixInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{MixInf}(\phi, \mathcal{D})$.*
 - (b) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{D})$ and*
 - i. *$x \in \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{D})$.*
 - ii. *$x \notin \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{MixSing}(\phi, \mathcal{D})$.*
7. *If $x \in \text{PureInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{PureInf}(\phi, \mathcal{D})$.*
 - (b) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{D})$ and*
 - i. *$x \in \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{D})$.*
 - ii. *$x \notin \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{PureSing}(\phi, \mathcal{D})$.*
8. *If $x \in \text{Sing}(\phi, \mathcal{C})$ then $x \in \text{Sing}(\phi, \mathcal{D})$.*
9. *If $x \in \text{NonSing}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{Reg}(\phi)$ implies $x \in \text{NonSing}(\phi, \mathcal{D})$.*
 - (b) *$x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D}) \cup \text{Nonapp}(\phi, \mathcal{D})$ implies $x \in \text{NonSing}(\phi, \mathcal{D})$.*
 - (c) *$x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D}) \cup \text{Nonapp}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{D})$.*

10. If $x \in \text{RemSing}(\phi, \mathcal{C})$ then
 - (a) $x \in \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{D})$.
 - (b) $x \notin \text{RemSing}(\phi, \mathcal{D})$ implies $x \in \text{EssSing}(\phi, \mathcal{D})$.
11. If $x \in \text{EssSing}(\phi, \mathcal{C})$ then $x \in \text{EssSing}(\phi, \mathcal{D})$.
12. If $x \in \text{MixSing}(\phi, \mathcal{C})$ then $x \in \text{MixSing}(\phi, \mathcal{D})$.
13. If $x \in \text{PureSing}(\phi, \mathcal{C})$ then $x \in \text{PureSing}(\phi, \mathcal{D})$.

Proof. The proofs follow from Propositions 16 and 17, Corollaries 18 and 19, the set equations given in the proof of Corollary 19 and the relevant definitions. \square

Figure 3, gives a graphic representation of Theorem 20. The solid arrows give the usual structure of the boundary classification, the dashed arrows show how the classification may change when \mathcal{C} is enlarged to \mathcal{D} , i.e. $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. Note that we have not included arrows based at a class that point to the same class.

The only surprising behaviour is that an essential point at infinity can become a removable singularity. This can only occur in a very specific set of circumstances, related to the difference between $\text{RemInf}(\phi)$ and $\text{RemSing}(\phi, \mathcal{C})$.

Lemma 21. *Let $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ be such that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. If $x \in \text{EssInf}(\phi, \mathcal{C})$ is such that $x \in \text{RemSing}(\phi, \mathcal{D})$ then for all $(\psi, U) \in \text{B}(M)$ so that $(\psi, U) \triangleright (\phi, \{x\})$ we know that,*

1. *there exists $p \in U$ so that $p \in \text{Irreg}(\psi) \cap (\text{App}_{\text{Inf}}(\psi, \mathcal{D}) \cup \text{Nonapp}(\psi, \mathcal{D}))$,*
2. *for all $\gamma \in \mathcal{D}$, bounded, so that $x \in \overline{\phi \circ \gamma}$ if $q \in U$ is such that $q \in \overline{\psi \circ \gamma}$ then $q \in \text{Reg}(\psi)$,*
3. $\text{Reg}(\psi) \cap U \neq \emptyset$.

Proof. Since $x \in \text{RemSing}(\phi, \mathcal{D}) = \text{Irreg}(\phi) \cap \text{App}_{\text{Sing}}(\phi, \mathcal{D}) \cap \text{RemSing}(\phi, \mathcal{D})$ we know that there exists $(\psi, U) \in \text{B}(M)$ so that $(\psi, U) \triangleright (\phi, \{x\})$ and $U \subset \text{NonSing}(\psi, \mathcal{D})$. If $U \subset \text{Reg}(\psi)$ then $x \in \text{NonSing}(\psi, \mathcal{C})$. This is a contradiction and therefore there exists $p \in U$ so that $p \in \text{Irreg}(\psi)$. As $p \in \text{NonSing}(\psi, \mathcal{D})$ we know that either $p \in \text{App}_{\text{Inf}}(\psi, \mathcal{D})$ or $p \in \text{Nonapp}(\psi, \mathcal{D})$. This proves item (1).

Since $x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ there exists $\gamma \in \mathcal{D}$, bounded, so that $x \in \overline{\phi \circ \gamma}$. By Theorem 17 of [7] there exists $q \in U$ so that $q \in \overline{\psi \circ \gamma}$. If $q \in \text{Irreg}(\psi)$ then $q \in \text{Sing}(\psi, \mathcal{D})$. This is a contradiction and therefore $q \in \text{Reg}(\psi)$. This proves items (2) and (3). \square

Thus in order for a boundary point $x \in \partial\phi(M)$, as given in the statement of Lemma 21, to exist, we know that, using the parametrization of curves given by \mathcal{D} , there exists at least one

1. bounded curve, converging to x , along which the metric behaves regularly. That is, the curves affine parameter is bounded, the Kretschmann scalar has a well defined limit, etc...
2. unbounded curve, converging to x ,

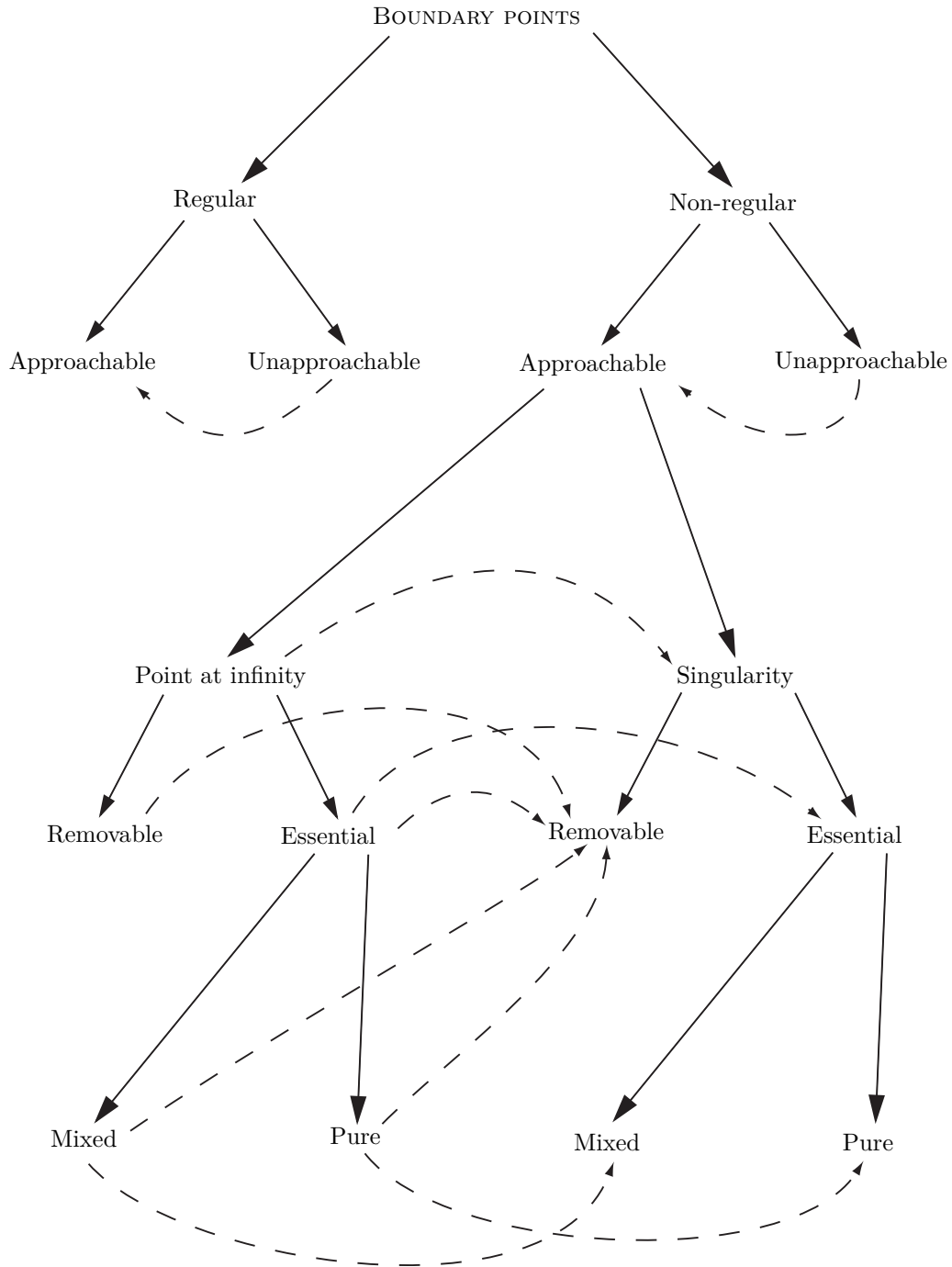


Figure 3: The changes to the boundary classification when $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. The solid arrows give the usual structure of the boundary classification, the dashed arrows show how the classification may change when \mathcal{C} is enlarged to \mathcal{D} . Note that we have not included arrows based at a class that point to the same class.

and the set of curves \mathcal{C} cannot contain any of the curves of the first type and must contain at least one curve of the second type while the set of curves \mathcal{D} must contain curves of both types. It is tempting therefore to attribute the existence of such points, in the classifications of \mathcal{C} and \mathcal{D} , to a surfeit of curves in the set \mathcal{C} . An example of this behaviour can be constructed from the directional singularity of the Curzon solution as it hides a portion of spacelike infinity, [13, 14].

3.2 Union

We now give a similar analysis for the union of $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$. Throughout this section we assume that $\mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ is well defined.

We will not go into as much detail as Section 3.1 for two reasons. First, the need to determine how the classification changes when taking the union of two b.p.p. satisfying sets is much less than that of adding additional curves to a b.p.p. satisfying set. Second, the first section provides an adequate example of how to determine additional detail if required.

Proposition 22. *Let $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ and let $\mathcal{E} = \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ then, for all $\phi \in \Phi(\mathcal{M})$,*

1. $\text{App}(\phi, \mathcal{C}) \cup \text{App}(\phi, \mathcal{D}) = \text{App}(\phi, \mathcal{E})$,
2. $\text{App}_{\text{Sing}}(\phi, \mathcal{C}) \cup \text{App}_{\text{Sing}}(\phi, \mathcal{D}) = \text{App}_{\text{Sing}}(\phi, \mathcal{E})$,
3. $\text{App}_{\text{Inf}}(\phi, \mathcal{E}) = \left(\text{App}_{\text{Inf}}(\phi, \mathcal{C}) - \text{App}_{\text{Sing}}(\phi, \mathcal{D}) \right) \cup \left(\text{App}_{\text{Inf}}(\phi, \mathcal{D}) - \text{App}_{\text{Sing}}(\phi, \mathcal{C}) \right)$,
4. $\text{Nonapp}(\phi, \mathcal{E}) = \text{Nonapp}(\phi, \mathcal{C}) \cap \text{Nonapp}(\phi, \mathcal{D})$,
5. $\text{NonSing}(\phi, \mathcal{E}) = \text{NonSing}(\phi, \mathcal{C}) \cap \text{NonSing}(\phi, \mathcal{D})$,
6. $\text{Rem}_{\text{Sing}}(\phi, \mathcal{E}) \subset \text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$,
7. *If $x \in (\text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ then for all $(\psi, U) \in \text{B}(M)$ so that $(\psi, U) \triangleright (\phi, \{x\})$ and $U \subset \text{NonSing}(\psi, \mathcal{C})$ ($U \subset \text{NonSing}(\psi, \mathcal{D})$) there exists $y \in U$ so that $y \in \text{Irreg}(\psi)$ and either $y \in \text{App}_{\text{Inf}}(\psi, \mathcal{C}) \cap \text{App}_{\text{Sing}}(\psi, \mathcal{D})$ ($y \in \text{App}_{\text{Inf}}(\psi, \mathcal{D}) \cap \text{App}_{\text{Sing}}(\psi, \mathcal{C})$) or $y \in \text{Nonapp}(\psi, \mathcal{C}) \cap \text{App}(\psi, \mathcal{D})$ ($y \in \text{Nonapp}(\psi, \mathcal{D}) \cap \text{App}(\psi, \mathcal{C})$),*
8. $\text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \cup \text{Ess}_{\text{Sing}}(\phi, \mathcal{D}) \subset \text{Ess}_{\text{Sing}}(\phi, \mathcal{E})$,
9. $\text{Ess}_{\text{Sing}}(\phi, \mathcal{E}) - (\text{Ess}_{\text{Sing}}(\phi, \mathcal{C}) \cup \text{Ess}_{\text{Sing}}(\phi, \mathcal{D})) = (\text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$.

Proof. Items (1), (2), follow from the construction of \mathcal{E} , see Definition 31 of [12]. Item (3) follows from items (1) and (2), the equation $(A \cup B) - C = (A - C) \cup (B - C)$ and as $B \subset A$ implies that $A - (B \cup C) = A - B - C$. Item (4) follows from item (1) and the definition. Item (5) follows from item (2), Definition 7 and the equation $A - (B \cup C) = (A - B) \cap (A - C)$. Item (6) follows directly from item (5) and Definition 15.

We now prove Item (7). Let $p \in (\text{Rem}_{\text{Sing}}(\phi, \mathcal{C}) \cap \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})) - \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ then for all $(\psi, U) \in \text{B}(M)$ so that $(\psi, U) \triangleright (\phi, \{p\})$ and $U \subset \text{NonSing}(\psi, \mathcal{C})$ we know that $U \not\subset \text{NonSing}(\psi, \mathcal{E})$. Thus there exists $q \in U$ so that $q \in \text{Irreg}(\psi)$, $q \notin \text{App}_{\text{Inf}}(\psi, \mathcal{E})$ and $q \notin \text{Nonapp}(\psi, \mathcal{E})$. Since $q \in \text{NonSing}(\psi, \mathcal{C})$ we know that $q \in \text{App}_{\text{Inf}}(\psi, \mathcal{C}) \cup \text{Nonapp}(\psi, \mathcal{C})$. If $q \in \text{App}_{\text{Inf}}(\psi, \mathcal{C})$ then $q \in \text{App}_{\text{Sing}}(\psi, \mathcal{D})$ (by item (3) and assumption), otherwise $q \in \text{NonSing}(\psi, \mathcal{E})$. If $q \in \text{Nonapp}(\psi, \mathcal{C})$ then $q \in \text{App}(\psi, \mathcal{D})$ (by item (4) and assumption), otherwise $q \in \text{NonSing}(\psi, \mathcal{E})$. This is sufficient to prove item (7).

Items (8) and (9) follow from item (6) and the relevant definitions. \square

Figures 4 and 5 give a graphical representation of Proposition 22. We skip the equivalent of Corollary 19, the missing details can be determined from Proposition 22.

Theorem 23. *Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ and let $\mathcal{E} = \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$. For all $\phi \in \Phi(\mathcal{M})$ we have that,*

1. *If $x \in \text{App}(\phi, \mathcal{C})$ then $x \in \text{App}(\phi, \mathcal{E})$*
2. *If $x \in \text{Nonapp}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{App}(\phi, \mathcal{D})$ implies $x \in \text{App}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{Nonapp}(\phi, \mathcal{D})$ implies $x \in \text{Nonapp}(\phi, \mathcal{E})$*
3. *If $x \in \text{Inf}(\phi, \mathcal{C})$ then*
 - (a) *$x \notin \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{Inf}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{E})$*
4. *If $x \in \text{RemInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \notin \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{RemInf}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{RemSing}(\phi, \mathcal{E})$*
5. *If $x \in \text{EssInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \notin \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{EssInf}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{E})$ and*
 - i. *$x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{RemSing}(\phi, \mathcal{E})$*
 - ii. *$x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{EssSing}(\phi, \mathcal{E})$*
6. *If $x \in \text{MixInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \notin \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{MixInf}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{E})$ and*
 - i. *$x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{RemSing}(\phi, \mathcal{E})$*
 - ii. *$x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{MixSing}(\phi, \mathcal{E})$*
7. *If $x \in \text{PureInf}(\phi, \mathcal{C})$ then*
 - (a) *$x \notin \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{PureInf}(\phi, \mathcal{E})$*
 - (b) *$x \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{E})$ and*
 - i. *$x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{RemSing}(\phi, \mathcal{E})$*
 - ii. *$x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{E})$ implies $x \in \text{PureSing}(\phi, \mathcal{E})$*
8. *If $x \in \text{Sing}(\phi, \mathcal{C})$ then $x \in \text{Sing}(\phi, \mathcal{E})$*
9. *if $x \in \text{NonSing}(\phi, \mathcal{C})$ then*
 - (a) *$x \in \text{Reg}(\phi)$ implies $x \in \text{NonSing}(\phi, \mathcal{E})$*
 - (b) *$x \notin \text{Reg}(\phi)$ and*

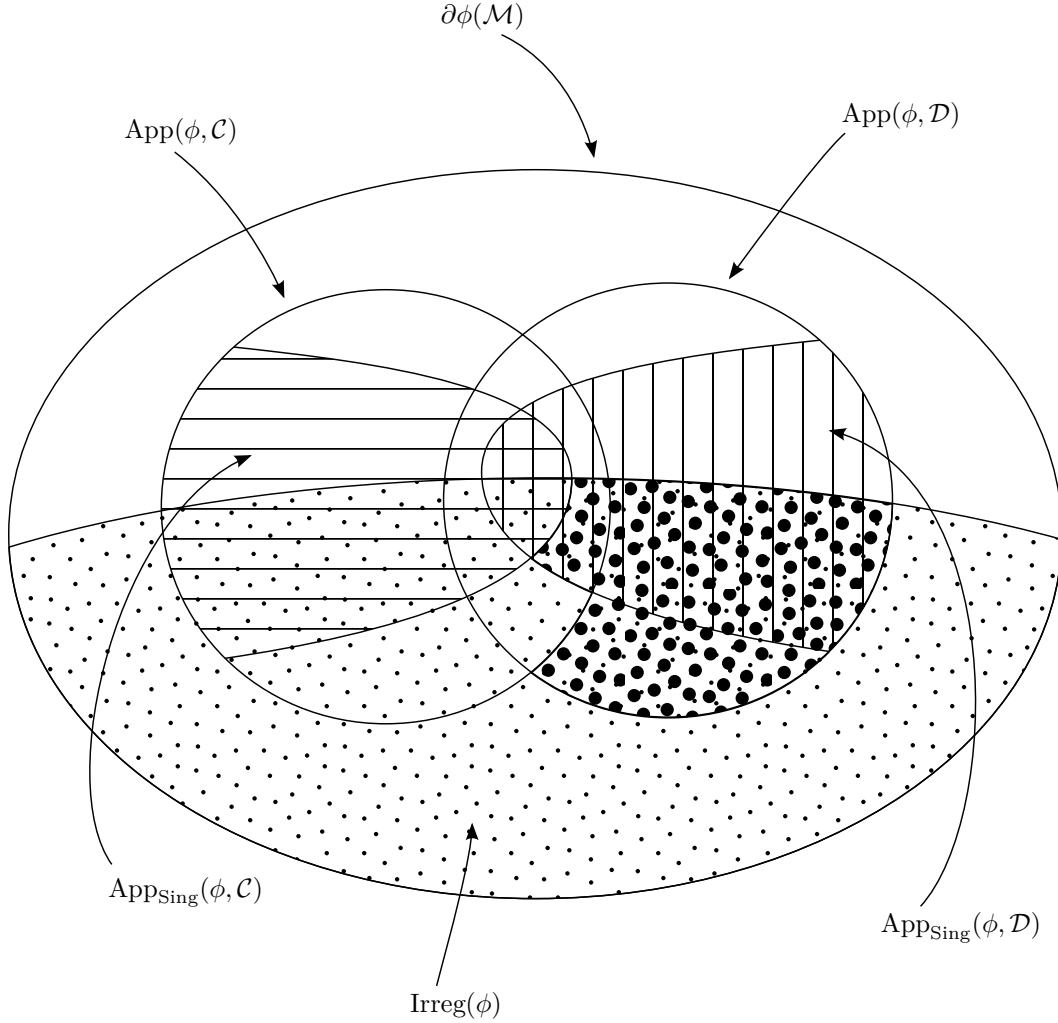


Figure 4: A venn diagram of $\partial\phi(\mathcal{M})$, $\text{Reg}(\phi)$, $\text{Irreg}(\phi)$, $\text{App}(\phi, \mathcal{C})$, $\text{AppSing}(\phi, \mathcal{C})$, $\text{AppInf}(\phi, \mathcal{C})$, $\text{App}(\phi, \mathcal{D})$, $\text{AppSing}(\phi, \mathcal{D})$, $\text{AppInf}(\phi, \mathcal{D})$, $\text{App}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$, $\text{AppSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$ and $\text{AppInf}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The largest oval is $\partial\phi(\mathcal{M})$. The dotted region, both small and large dots, is $\text{Irreg}(\phi)$. The left circle is $\text{App}(\phi, \mathcal{C})$, the right circle is $\text{App}(\phi, \mathcal{D})$. The horizontally ruled region is $\text{AppSing}(\phi, \mathcal{C})$, the region of the left circle not ruled by horizontal lines is $\text{AppInf}(\phi, \mathcal{C})$. Likewise, the vertically ruled region is $\text{AppSing}(\phi, \mathcal{D})$ and the region of the right circle not ruled by vertical lines is $\text{AppInf}(\phi, \mathcal{D})$. The union of both circles is $\text{App}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The region that is ruled, either by horizontal lines or vertical lines or both is $\text{AppSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The region inside the union of the circles that is not ruled is $\text{AppInf}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The region covered by large dots is $\text{Irreg}(\phi) \cap \text{Nonapp}(\phi, \mathcal{C}) \cap \text{App}(\phi, \mathcal{D}) \cup \text{Irreg}(\phi) \cap \text{AppInf}(\phi, \mathcal{C}) \cap \text{AppSing}(\phi, \mathcal{D})$, see item (7) of Proposition 22. Note that the sets mentioned in brackets in item (7) of Proposition 22 can be found by interchanging \mathcal{C} and \mathcal{D} in the labels of this diagram.

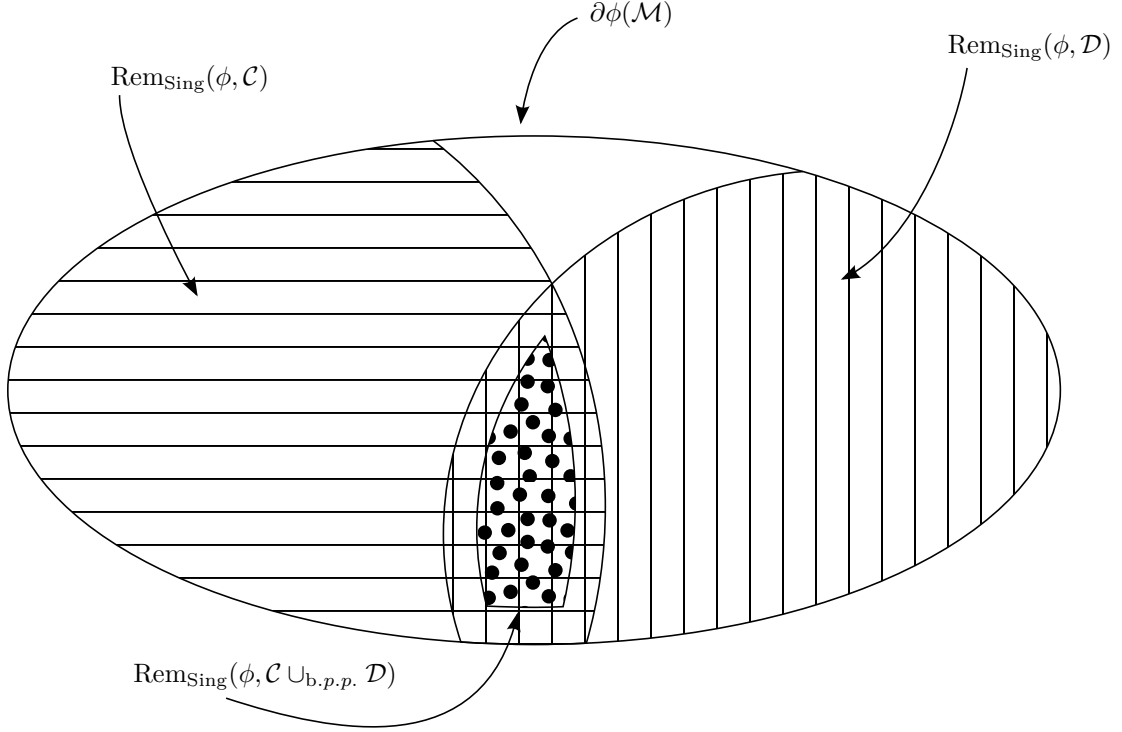


Figure 5: A venn diagram of $\partial\phi(\mathcal{M})$, $\text{RemSing}(\phi, \mathcal{C})$, $\text{EssSing}(\phi, \mathcal{C})$, $\text{RemSing}(\phi, \mathcal{D})$, $\text{EssSing}(\phi, \mathcal{D})$, $\text{RemSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$ and $\text{EssSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The oval is $\partial\phi(\mathcal{M})$. The horizontally ruled region is $\text{RemSing}(\phi, \mathcal{C})$, the region not ruled by horizontal lines is $\text{EssSing}(\phi, \mathcal{C})$. Likewise, the vertically ruled region is $\text{RemSing}(\phi, \mathcal{D})$ and the region not ruled by vertical lines is $\text{EssSing}(\phi, \mathcal{D})$. The dotted region is $\text{RemSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$ and the region that is not dotted is $\text{EssSing}(\phi, \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D})$. The region that is ruled by both horizontal and vertical lines, but is not dotted consists of those boundary points that are described by item (7) of Proposition 22, see Figure 4.

- i. $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{NonSing}(\phi, \mathcal{E})$
- ii. $x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $x \in \text{Sing}(\phi, \mathcal{E})$

10. If $x \in \text{RemSing}(\phi, \mathcal{C})$ then

- (a) $x \in \text{RemSing}(\phi, \mathcal{E})$ implies $x \in \text{RemSing}(\phi, \mathcal{E})$
- (b) $x \notin \text{RemSing}(\phi, \mathcal{E})$ implies $x \in \text{EssSing}(\phi, \mathcal{E})$

11. If $x \in \text{EssSing}(\phi, \mathcal{C})$ then $x \in \text{EssSing}(\phi, \mathcal{E})$

12. If $x \in \text{MixSing}(\phi, \mathcal{C})$ then $x \in \text{MixSing}(\phi, \mathcal{E})$

13. If $x \in \text{PureSing}(\phi, \mathcal{C})$ then $x \in \text{PureSing}(\phi, \mathcal{E})$.

Membership of $\text{RemSing}(\phi, \mathcal{E})$ can be checked using item (7) of Proposition 22. The same statements hold when interchanging \mathcal{C} and \mathcal{D} .

Proof. Each item follows from Propositions 16 and 22. \square

We again see the surprising behaviour that essential points at infinity can become removable singularities. As before, this can only occur in a specific set of circumstances, namely those described in item (7) of Proposition 22 and by Lemma 21 (where we consider $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{E}$).

An example of this, fitting the situation of item (7) of Proposition 22, can be constructed using the directional singularity of the Curzon solution, [13, 14], where \mathcal{C} contains curves classifying the directional singularity as a point of spacelike infinity and \mathcal{D} contains curves classifying the directional singularity as a singularity.

4 Changes to the abstract boundary classification as we change \mathcal{C}

Now that we know how the classification of boundary points changes we can determine how the classification of abstract boundary points changes.

4.1 Subset

Theorem 24. Let $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ so that $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$ then,

1. If $[(\phi, \{x\})] \in \text{APP}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{APP}(\mathcal{D})$
2. If $[(\phi, \{x\})] \in \text{NONAPP}(\mathcal{C})$ then either $[(\phi, \{x\})] \in \text{APP}(\mathcal{D})$ or $[(\phi, \{x\})] \in \text{NONAPP}(\mathcal{D})$.
3. If $[(\phi, \{x\})] \in \text{INDET}(\mathcal{C})$ then
 - (a) $x \in \text{Reg}(\phi)$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - (b) $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{C})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - (c) $x \in \text{App}_{\text{Sing}}(\phi, \mathcal{C})$ and
 - i. $x \in \text{RemSing}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - ii. $x \notin \text{RemSing}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{SING}(\mathcal{D})$.

4. If $[(\phi, \{x\})] \in \text{INF}(\mathcal{C})$ then
 - (a) $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{INF}(\mathcal{D})$
 - (b) $x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ and
 - i. $x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - ii. $x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{SING}(\mathcal{D})$.
5. If $[(\phi, \{x\})] \in \text{MIXINF}(\mathcal{C})$ then
 - (a) $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{MIXINF}(\mathcal{D})$
 - (b) $x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ and
 - i. $x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - ii. $x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{D})$.
6. If $[(\phi, \{x\})] \in \text{PUREINF}(\mathcal{C})$ then
 - (a) $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{PUREINF}(\mathcal{D})$
 - (b) $x \notin \text{App}_{\text{Inf}}(\phi, \mathcal{D})$
 - i. $x \in \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then $[(\phi, \{x\})] \in \text{INDET}(\mathcal{D})$
 - ii. $x \notin \text{Rem}_{\text{Sing}}(\phi, \mathcal{D})$ then $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{D})$.
7. If $[(\phi, \{x\})] \in \text{SING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{SING}(\mathcal{D})$
8. If $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{D})$
9. If $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{D})$

Proof. This follows from Definitions 8, 9, 10 and 11 and from Theorem 20. Note that $\text{Reg}(\phi), \text{Inf}(\phi, \mathcal{C}) \subset \text{Rem}_{\text{Sing}}(\phi, \mathcal{C})$ since elements of $\text{Reg}(\phi)$ and $\text{Inf}(\phi, \mathcal{C})$ are covered by themselves. \square

As before we give a graphical depiction of these results in Figure 6. The solid arrows give the usual structure of the abstract boundary classification, the dashed arrows show how the classification may change when \mathcal{C} is enlarged to \mathcal{D} , i.e. $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. Note that we have not included arrows based at a class that point to the same class.

4.1.1 Union

Theorem 25. Let $\mathcal{C}, \mathcal{D} \in \text{BPP}(\mathcal{M})$ so that $\mathcal{E} = \mathcal{C} \cup_{\text{b.p.p.}} \mathcal{D}$ is well defined, then,

1. If $[(\phi, \{x\})] \in \text{APP}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{APP}(\mathcal{E})$.
2. If $[(\phi, \{x\})] \in \text{NONAPP}(\mathcal{C})$ then
 - (a) $[(\phi, \{x\})] \in \text{APP}(\mathcal{D})$ implies $[(\phi, \{x\})] \in \text{APP}(\mathcal{E})$
 - (b) $[(\phi, \{x\})] \in \text{NONAPP}(\mathcal{D})$ implies $[(\phi, \{x\})] \in \text{NONAPP}(\mathcal{E})$.
3. If $[(\phi, \{x\})] \in \text{INDET}(\mathcal{C})$ then
 - (a) $x \in \text{Reg}(\phi)$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$
 - (b) $x \in \text{App}_{\text{Inf}}(\phi, \mathcal{C})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$

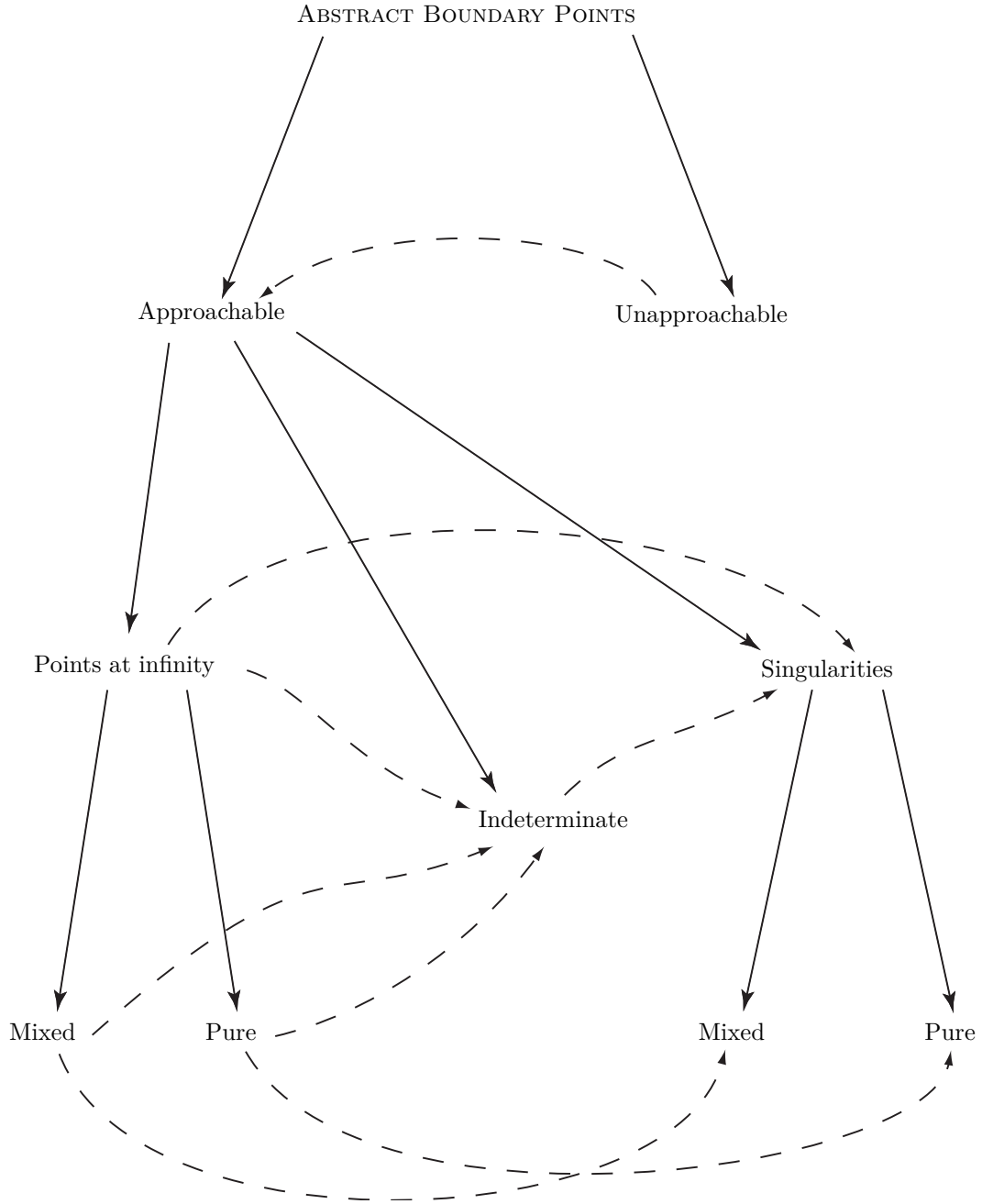


Figure 6: The changes to the abstract boundary classification when $\mathcal{C} \subset_{\text{b.p.p.}} \mathcal{D}$. The solid arrows give the usual structure of the boundary classification, the dashed arrows show how the classification may change between \mathcal{C} and \mathcal{D} . Note that we have not included arrows based at a class that point to the same class.

- (c) $x \in \text{AppSing}(\phi, \mathcal{C})$ and
 - i. $x \in \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$
 - ii. $x \notin \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{SING}(\mathcal{E})$.
- 4. If $[(\phi, \{x\})] \in \text{INF}(\mathcal{C})$ then
 - (a) $x \notin \text{AppSing}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{INF}(\mathcal{E})$
 - (b) $x \in \text{AppSing}(\phi, \mathcal{D})$ and
 - i. $x \in \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$
 - ii. $x \notin \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{SING}(\mathcal{E})$.
- 5. If $[(\phi, \{x\})] \in \text{MIXINF}(\mathcal{C})$ then
 - (a) $x \notin \text{AppSing}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{MIXINF}(\mathcal{E})$
 - (b) $x \in \text{AppSing}(\phi, \mathcal{D})$ and
 - i. $x \in \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$
 - ii. $x \notin \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{E})$.
- 6. If $[(\phi, \{x\})] \in \text{PUREINF}(\mathcal{C})$ then
 - (a) $x \notin \text{AppSing}(\phi, \mathcal{D})$ implies $[(\phi, \{x\})] \in \text{PUREINF}(\mathcal{E})$
 - (b) $x \in \text{AppSing}(\phi, \mathcal{D})$ and
 - i. $x \in \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{INDET}(\mathcal{E})$
 - ii. $x \notin \text{RemSing}(\phi, \mathcal{E})$ implies $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{E})$.
- 7. If $[(\phi, \{x\})] \in \text{SING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{SING}(\mathcal{E})$
- 8. If $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{MIXSING}(\mathcal{E})$
- 9. If $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{C})$ then $[(\phi, \{x\})] \in \text{PURESING}(\mathcal{E})$.

Membership of $\text{RemSing}(\phi, \mathcal{E})$ can be checked using item (7) of Proposition 22. The same statements hold when interchanging \mathcal{C} and \mathcal{D} .

Proof. This follows from Definitions 8, 9, 10 and 11 and from Theorem 23. □

5 Acknowledgements

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